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Distortion of Biholomorphic Maps on Homogeneous Domains in J^*-Algebras

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1. INTRODUCTION

Several authors (R. W. Barnard, C. H. Fitzgerald, and S. Gong [2], P. Duren and W. Rudin [6], Ky Fan [7], S. Gong, S. K. Wang, and Q. H. Yu [8], S. Gong and H. A. Zheng [9], K. T. Hahn [10], L. A. Harris [13], M. Hervé [15], Y. Q. Huang [16], X. Liu [18], X. L. Zhang [26], and others) studied (with some restrictions—see the Appendix) the higher dimensional versions of lower bounds in the following classical Koebe inequalities (see, e.g., P. Koebe [17], R. Nevanlinna [20], and Ch. Pommerenke [19, p. 22])

$$(1-|u|^2)|DF(u)|/4 \le \operatorname{dist}[F(u), \partial F(\Delta)] \le (1-|u|^2)|DF(u)|, \quad (1)$$

$u \in \Delta$, $F\colon \Delta \to \mathbb{C}$—conformal maps, $\Delta = \{x \in \mathbb{C}\colon |x| < 1\}$. The results of our paper provide versions of upper bounds in the above inequalities for arbitrary biholomorphic maps in the generalized right half-planes and in bounded symmetric homogeneous domains of infinite dimensional complex Banach spaces called J^*-algebras.

2. DEFINITIONS, NOTATIONS, AND STATEMENT OF RESULTS

Let H and K be Hilbert spaces over $\mathbb{C}$, let $L(H, K)$ denote the Banach space of all bounded linear operators X from H to K with the operator norm, and let $\mathfrak{B} \subset L(H, K)$ be a J^*-algebra, i.e., a closed complex linear subspace of $L(H, K)$ such that $XX^*X \in \mathfrak{B}$ whenever $X \in \mathfrak{B}$.

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Let

$$\mathfrak{B}_0 = \{X \in \mathfrak{B} \colon \|X\| < 1\}$$

and, if a J^*-algebra $\mathfrak{B}$ contains a nonzero partial isometry V, let

$$\mathfrak{M}_V = \left\{X \in \mathfrak{B} \colon 2\,\mathrm{Re}\,V^*X - X^*(I_K - VV^*)X + I_H - V^*V > 0\right\}.$$

For an isometry $V \in \mathfrak{B}$, let

$$\mathfrak{R}_V = \left\{X \in \mathfrak{B} \colon \mathrm{Re}\,V^*X - X^*(I_K - VV^*)X > 0\right\}.$$

Here I_H and I_K denote the identity maps on H and K, respectively, and $\mathrm{Re}\,X$ denotes the real part of an operator $X \in L(H,H)$, i.e., $\mathrm{Re}\,X = (1/2)(X + X^*)$.

For $X \in \mathfrak{M}_V$, let

$$A_X(\mathfrak{M}_V) = A_V^{1/2}(I_H + X^*V)^{-1}(\mathrm{RE}\,V^*X)(I_H + V^*X)^{-1}A_V^{1/2},$$
$$B_X(\mathfrak{M}_V) = B_V^{1/2}(I_K + XV^*)^{-1}(\mathrm{RE}\,XV^*)(I_K + VX^*)^{-1}B_V^{1/2},$$

where

$$A_V = I_H + V^*V, \qquad B_V = I_K + VV^*,$$
$$\mathrm{RE}\,V^*X = 2\,\mathrm{Re}\,V^*X - X^*(I_K - VV^*)X + I_H - V^*V,$$
$$\mathrm{RE}\,XV^* = 2\,\mathrm{Re}\,XV^* - X(I_H - V^*V)X^* + I_K - VV^*.$$

For $X \in \mathfrak{R}_V$, let

$$A_X(\mathfrak{R}_V) = (I_H + X^*V)^{-1}(\mathrm{RE}\,V^*X)(I_H + V^*X)^{-1},$$
$$B_X(\mathfrak{R}_V) = (I_K + XV^*)^{-1}(\mathrm{RE}\,XV^*)(I_K + VX^*)^{-1},$$

where

$$\mathrm{RE}\,V^*X = \mathrm{Re}\,V^*X - X^*(I_K - VV^*)X,$$
$$\mathrm{RE}\,XV^* = \mathrm{Re}\,XV^* - X(I_H - V^*V)X^*.$$

For $X \in \mathfrak{B}_0$, let

$$A_X = I_H - X^*X \qquad \text{and} \qquad B_X = I_K - XX^*.$$

We prove the following

THEOREM 2.1. (a) *Let* $\mathfrak{B} \subset \mathbf{L}(H, K)$ *be a* J^**-algebra containing a nonzero partial isometry* V. *If* $F\colon \mathfrak{M}_V \to \mathfrak{B}$ *is a biholomorphic map and* $X \in \mathfrak{M}_V$, *then*

$$\begin{aligned}\operatorname{dist}\left[F(X), \partial F(\mathfrak{M}_V)\right] & \\ \leq \|DF(X)\| \, & \left\|A_V^{-1/2}(I_H + V^*X)\right\| \left\|(I_K + XV^*)B_V^{-1/2}\right\| \\ & \times \left(\|A_X(\mathfrak{M}_V)\| \, \|B_X(\mathfrak{M}_V)\|\right)^{1/2}.\end{aligned}$$

(b) *Let* $\mathfrak{B} \subset \mathbf{L}(H, K)$ *be a* J^**-algebra containing an isometry* V. *If* $F\colon \mathfrak{R}_V \to \mathfrak{B}$ *is a biholomorphic map and* $X \in \mathfrak{R}_V$, *then*

$$\begin{aligned}\operatorname{dist}\left[F(X), \partial F(\mathfrak{R}_V)\right] \leq \|DF(X)\| \, \|I_H + V^*X\| & \\ \times \|I_K + XV^*\| \left(\|A_X(\mathfrak{R}_V)\| \, \|B_X(\mathfrak{R}_V)\|\right)^{1/2}&.\end{aligned}$$

(c) *Let* $\mathfrak{B} \subset \mathbf{L}(H, K)$ *be a* J^**-algebra. If* $F\colon \mathfrak{B}_0 \to \mathfrak{B}$ *is a biholomorphic map and* $X \in \mathfrak{B}_0$, *then*

$$\operatorname{dist}\left[F(X), \partial F(\mathfrak{B}_0)\right] \leq \|DF(X)\| \, \|A_X^{1/2}\| \, \|B_X^{1/2}\|.$$

Remark 2.1. J^*-algebras, being natural generalization of C^*-algebras, B^*-algebras, JC^*-algebras, ternary algebras, complex Hilbert spaces, and others, are infinite dimensional complex Banach spaces whose open unit balls $\mathfrak{B}_0$ are bounded symmetric homogeneous domains. In particular, all four types of the classical Cartan bounded symmetric homogeneous domains [5] and their infinite dimensional analogues [11, 12, 14] are the open unit balls $\mathfrak{B}_0$ in some J^*-algebras $\mathfrak{B}$. The unbounded convex domains $\mathfrak{M}_V$ and $\mathfrak{R}_V$ play the role of the generalized right half-planes [23, 24].

3. PROOF OF THEOREM 2.1

(a) We have

$$\begin{aligned}VA_V^{-1/2} = B_V^{-1/2}V = 2^{-1/2}V, \qquad A_V^{-1} = I_H - 2^{-1}V^*V,& \\ B_V^{-1} = I_K - 2^{-1}VV^*.&\end{aligned} \tag{2}$$

Thus, if we denote

$$S = f_V^{-1}(X) = B_V^{-1/2}(X - V)(I_H + V^*X)^{-1} A_V^{1/2}, \qquad X \in \mathfrak{M}_V, \quad (3)$$

where f_V is a biholomorphic map of $\mathfrak{B}_0$ and $\mathfrak{M}_V$ (cf. [23, Theorem 5, p. 499]), then we obtain (see [25, p. 252])

$$A_S = I_H - S^*S = A_X(\mathfrak{M}_V) \quad (4)$$

and since, by (2),

$$(X - V)(I_H + V^*X)^{-1} = B_V(I_K + XV^*)^{-1}(X - V)A_V^{-1}, \quad (5)$$

we also get

$$B_S = I_K - SS^* = B_X(\mathfrak{M}_V). \quad (6)$$

Hence, in particular, after taking account of $B_S > 0$, since $S \in \mathfrak{B}_0$, we have

$$\mathrm{RE}\ XV^* > 0.$$

Let $S \in \mathfrak{B}_0$ be arbitrary and fixed and let G_S: $\mathfrak{B}_0 \to \mathfrak{B}$ be a map defined by the formula

$$G_S(Z) = (F \circ f_V \circ T_S)(Z) - (F \circ f_V)(S), \qquad Z \in \mathfrak{B}_0, \quad (7)$$

where T_S is a Möbius biholomorphic map of $\mathfrak{B}_0$ onto $\mathfrak{B}_0$ defined by the formula (see L. A. Harris [11, Theorem 2, p. 20])

$$T_S(Z) = B_S^{-1/2}(Z + S)(I_H + S^*Z)^{-1} A_S^{1/2}, \qquad Z \in \mathfrak{B}_0.$$

Since

$$\begin{aligned} DG_S(0)(P) &= DF[(f_V \circ T_S)(0)]\{Df_V[T_S(0)][DT_S(0)(P)]\} \\ &= DF[f_V(S)]\{Df_V(S)[DT_S(0)(P)]\}, \end{aligned}$$

$P \in \mathfrak{B}$, therefore, by using the inverse map theorem [21], the map W_S: $\mathfrak{B}_0 \to \mathfrak{B}$ defined by the formula

$$W_S(Z) = [DT_S(0)]^{-1}\Big\{[Df_V(S)]^{-1}\Big\{[DF(f_V(S))]^{-1}\{[G_S(Z)]\}\Big\}\Big\}, \qquad Z \in \mathfrak{B}_0, \quad (8)$$

is biholomorphic and satisfies the conditions

$$W_S(0) = 0 \qquad \text{and} \qquad DW_S(0) = I_{\mathfrak{B}}. \quad (9)$$

Now, let us observe that (see [23, p. 510])

$$Df_V(S)(P) = B_V^{1/2}(I_K - SV^*)^{-1}P(I_H - V^*S)^{-1}A_V^{1/2}, \tag{10}$$

$A_S > 0$, $B_S > 0$, and since (see [23, p. 507])

$$DT_S(0)(P) = B_S^{1/2}PA_S^{1/2} \qquad \text{for } P \in \mathfrak{B}, \tag{11}$$

from (8), (10), and (11) we get

$$\begin{aligned}\|G_S(Z)\| &\le \|DF[f_V(S)]\| \, \|Df_V(S)\{DT_S(0)[W_S(Z)]\}\| \\ &\le \|DF[f_V(S)]\| C(V,S) \|A_S\|^{1/2} \|B_S\|^{1/2} \|W_S(Z)\|, \quad Z \in \mathfrak{B}_0,\end{aligned}$$

where

$$C(V,S) = \left\| B_V^{1/2}(I_K - SV^*)^{-1} \right\| \, \left\| (I_H - V^*S)^{-1}A_V^{1/2} \right\|$$

and, consequently, from (7) and (9) we get

$$\begin{aligned}&\operatorname{dist}\big[(F\circ f_V)(S), \partial(F\circ f_V)(\mathfrak{B}_0)\big] \\ &\quad \le \|DF[f_V(S)]\| C(V,S) \|A_S\|^{1/2} \|B_S\|^{1/2} \operatorname{dist}\big[0, \partial W_S(\mathfrak{B}_0)\big].\end{aligned} \tag{12}$$

Let $\mu = \operatorname{dist}[0, \partial W_S(\mathfrak{B}_0)]$. By M. Hervé [15] and L. A. Harris [13], we have $\mu > 0$. Moreover, $\mu < \infty$; otherwise, by the Liouville theorem for holomorphic maps in complex Banach spaces, the map W_S^{-1} is constant, a contradiction. Now, define the map $M_S\colon \mathfrak{B}_0 \to \mathfrak{B}_0$ by the formula $M_S(Z) = W_S^{-1}(\mu Z)$. Since $M_S(0) = 0$, by the Schwarz lemma, we obtain

$$\|\mu P\| = \|DM_S(0)(P)\| \le \|P\|, \qquad P \in \mathfrak{B},$$

or, equivalently,

$$\mu \le 1. \tag{13}$$

If $S = f_V^{-1}(X)$, $X \in \mathfrak{M}_V$, then, using (3), (5), and (2), we obtain

$$\begin{aligned}I_K - SV^* &= B_V^{1/2}(I_K + XV^*)^{-1}B_V^{1/2}, \\ I_H - V^*S &= A_V^{1/2}(I_H + V^*X)^{-1}A_V^{1/2}\end{aligned}$$

and, consequently,

$$C(V,S) = \left\| (I_K + XV^*)B_V^{-1/2} \right\| \, \left\| A_V^{-1/2}(I_H + V^*X) \right\|. \tag{14}$$

Finally, from (12), using (4), (6), (13), and (14), we get the assertion.

(b) The biholomorphic map f_V of $\mathfrak{B}_0$ onto $\mathfrak{R}_V$ is defined by the formula

$$f_V(X) = (X + V)(I_H - V^*X)^{-1}, \qquad X \in \mathfrak{B}_0, \tag{15}$$

and, moreover,

$$f_V^{-1}(X) = B_V^{-1}(X - V)(I_H + V^*X)^{-1} A_V, \qquad X \in \mathfrak{R}_V.$$

Therefore, using arguments similar to those given in part (a) where f_V defined by (3) is replaced by that from (15), we immediately get the assertion.

(c) Using analogous considerations as in the proof of part (b) where the map f_V defined by (15) is replaced by the identity, we obtain the assertion.

4. APPENDIX

Since 1910 and, especially, after 1970 the concepts of geometrical function theory of one complex variable have been well established, and some higher-dimensional complex cases (in particular, the concepts of H. Cartan [4] concerning biholomorphic convex and starlike maps) have been solved. Interesting results on convex and starlike maps and the subordination principle in complex Banach spaces were established by T. J. Suffridge [22]. There are also important questions related to them.

Let E and F be complex Banach spaces such that $\dim_{\mathbb{C}}(E), \dim_{\mathbb{C}}(F) > 1$, and let E_0 denote the open unit ball in E. Let $\mathcal{S}$ denote the class of all biholomorphic maps $f\colon E_0 \to F$ of the form

$$f(x) = \sum_{m=1}^{\infty} (1/m!) D^m f(0)(x^m), \qquad x \in E_0.$$

From [23, Theorem 3b, p. 498] we have the following

THEOREM 4.1. *If $f \in \mathcal{S}$ and $f(E_0)$ is convex, then*

$$\|(1/m!) D^m f(0)(x^m)\| \le \|Df(0)\| \, \|x\|^m$$

for $m = 2, 3, \ldots$ and $x \in E_0$.

The above result and the theorem of L. De Branges [3] (see also [1, 4]) inspire the following

Problem 4.1. Let $f \in \mathcal{S}$ and let $f(E_0)$ be starlike. Are the inequalities

$$\|(1/m!) D^m f(0)(x^m)\| \le m \|Df(0)\| \, \|x\|^m \tag{16}$$

for $m = 2, 3, \ldots$ and $x \in E_0$ true?

The affirmative answer to Problem 4.1 for $m = 2$ can be given if we use the following property of f due to T. J. Suffridge [22, Theorem 1, p. 150]: then

$$f(x) = Df(x)(w(x)), \tag{17}$$

where $w\colon E_0 \to E$ is a holomorphic map such that $w(0) = 0$, $Dw(0) = I_E$, and $\operatorname{Re}(l \circ w)(x) > 0$ for $x \in E_0 \setminus \{0\}$ and $l \in E'$ satisfying $l(x) = \|x\|$ and $\|l\| = 1$.

Indeed, from (17), putting $ux, 0 < |u| < 1$, instead of x, dividing by u, differentiating with respect to u, passing to the limit with $u \to 0$, and multiplying by 2!, we obtain

$$\| D^2 f(0)(x^2) \| \leq \| Df(0) \| \, \| D^2 w(0)(x^2) \|, \qquad x \in E_0.$$

This implies (16) for $m = 2$ since $\|D^2 w(0)(x^2)\| \leq 2\|Dw(0)\| \, \|x\|^2 = 2\|x\|^2$ in virtue of the Hahn-Banach theorem and the inequality $|c_2| \leq 2$ for maps with positive real part in $\mathbb{C}$ (see [19, Corollary 2.3, p. 41]).

Problem 4.1 when $E = F = \mathbb{C}^2$, $f \in S$, and the set $f(E_0)$ is not starlike has a negative solution. Namely, for an arbitrary and fixed $m = 2, 3, \ldots,$ there exists a map $f \in S$ of the form

$$f(x_1, x_2) = (x_1 + bx_2^m, x_2), \qquad \text{where } |b| > m,$$

such that $\|Df(0)\| = 1$ and $\|(1/m!) D^m f(0)\| = |b|$. The above triangular map is also an example of that, in this case, the counterpart of lower bounds in inequalities (1) does not hold, either.

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